Computational Procedures for Recent Analyses of Counterflow Heat Exchangers

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Recently, an orthogonal expansion technique was used to describe the temperature distribution in laminar flow double-pipe heat exchangers (*2*). One of the difficulties with this approach is the calculation of the expansion coefficients, since they cannot be determined by standard techniques. A computational scheme for obtaining the expansion coefficients was described in this paper, and the expansion coefficients thus calculated were used to obtain the effectiveness for counterflow exchangers.

In a later issue of the *Journal* Stein (*4*) presented an alternate computational scheme for solving the same system of equations for the expansion coefficients. Called the *Argonne procedure* by Stein, this scheme, along with the initial one, named by Stein the *Nunge-Gill procedure*, was applied to a countercurrent plug flow system. From a comparison of the results, Stein concluded that the Nunge-Gill procedure fails to give accurate numerical values under certain conditions in the plug flow case. Furthermore, Stein states on page 1219, "The reader is invited to study the results shown in these tables as related to the treatment of Equation (22) of reference *2* and Equation (38) in reference *4* as well as the tabulations of the expansion coefficients in references *2* and *3*." This implies that one can extrapolate the plug flow case, which has a finite discontinuity in the velocity profile, to estimate the accuracy of the laminar flow calculations.

In contrast, we will show that the two computational schemes yield essentially the same effectiveness for those cases considered in references *1*, *2*, and *2a*. Also, some of the local temperature distributions and Nusselt numbers given in these references were compared with finite-difference calculations, and in all cases the comparisons were favorable (*1*, *2a*).

Thus, the purpose of this communication is to present a comparison of the effectiveness of the two computational schemes for the laminar flow case, a comparison similar to that made by Stein, for the plug flow exchanger. We shall restrict our attention to those results which are already in the literature (*2*) and cover the range $0.8 \leq KH \leq 4.5$. For the comparison, the problem of calculating the exchanger effectiveness, which requires the computation of the expansion coefficients, was reworked for laminar countercurrent flow in a concentric tube exchanger according to the Argonne procedure. The main conclusion to be drawn from these calculations is that, for the laminar flow cases studied previously, both procedures yield essentially the same effectiveness and that our original results (*2*) are accurate enough for the practical calculation of effectiveness, temperature distributions, and Nusselt numbers.

DISCUSSION OF SERIES CONVERGENCE

Because of the form of the equations which must be solved for the expansion coefficients, for example, Equation (38) of reference *2*, the addition of successive terms to the series can affect the lower-order expansion coefficients. The convergence and accuracy of the results can be judged by inspection of the results of successive calculations, a converged value of the expansion coefficient being obtained when the value is independent of the number of terms carried in the expansion. For Stein's plug flow system it is straightforward to obtain as many eigenvalues as desired, since for this system the eigenfunctions are given in closed form as cosines and hyperbolic cosines. In laminar flow systems one must resort to numerical techniques to find the eigenfunctions, and the higher-order functions are difficult to obtain accurately because of their periodic nature. That is, the periodicity increases in direct proportion to the order, since an n^{th} order eigenfunction crosses the axis n times. Thus, to maintain roughly the same accuracy in calculating eigenfunctions numerically, one must increase the number of increments spanning the field of integration in proportion to the order. This markedly increases the computing time involved.

Because of the practical difficulties associated with calculating the higher-order eigenfunctions, an alternate check of the results, in the form of a finite-difference solution, was provided by Nunge and Gill. It was found that six eigenfunctions were sufficient to give good agreement between the two solutions. Hence in the cases studied, because of the numerical difficulties and in view of the agreement with the finite-difference results, it was practical to limit the maximum number of terms to six. Illustrations of the agreement between the finite-difference and orthogonal-expansion techniques were given in Figures 6

to 8 of reference *1*, and additional comparisons were provided in Figures 8 to 11 of reference *2a*. In each case the agreement was good. These results suggest that the higher-order expansion coefficients reported are sufficiently accurate for most practical purposes, since the local temperature distributions and the Nusselt numbers calculated by using them are in good agreement with the finite-difference results.

COMPARISON OF EXCHANGER EFFECTIVENESS

Since the exchanger effectiveness is the most useful result for exchanger design, we shall restrict our attention here to comparisons of this quantity.

Tabulation of all the expansion coefficients obtained by both methods is impractical because of the enormous number of data involved. Comparisons made seemed to show that the Argonne procedure converges monotonically and more rapidly to the expansion coefficients than that reported in reference *2*. However, the results given (*1* to *2a*) appear to be sufficiently accurate for practical purposes, as discussed above for local temperature and Nusselt number distributions and as shown below for the effectiveness.

Table 1. Comparison of Effectiveness Values Calculated by Numerical Procedures

M	H	K	K_w	L or Z	ϵ_A	ϵ_{N-G}	ϵ_o	ϵ_L
4(4)	0.20	4.0	0.0	0.05	0.485	0.465	0.461	0.509
4(4)	0.20	4.0	0.0	0.10	0.700	0.724	0.676	0.724
4(4)	0.20	4.0	0.2	0.05	0.438	0.426	0.423	0.453
4(4)	0.20	4.0	0.2	0.10	0.651	0.638	0.636	0.666
4(4)	0.20	4.0	0.2	0.50	1.003	0.988	0.988	1.018
5(4)	0.50	2.0	0.0	0.05	0.329	0.335	0.329	0.329
5(4)	0.50	2.0	0.0	0.10	0.501	0.504	0.500	0.502
4(4)	0.50	2.0	0.0	0.50	0.918	0.917	0.917	0.919
5(4)	0.50	2.0	0.2	0.05	0.283	0.287	0.283	0.282
5(4)	0.50	2.0	0.2	0.10	0.444	0.448	0.445	0.445
4(4)	0.50	2.0	0.2	0.50	0.885	0.885	0.884	0.886
5(4)	0.50	4.0	0.0	0.05	0.237	0.241	0.235	0.239
5(4)	0.50	4.0	0.0	0.10	0.372	0.375	0.370	0.374
4(4)	0.50	4.0	0.0	0.50	0.816	0.815	0.814	0.818
5(4)	0.50	4.0	1.0	0.05	0.155	0.151	0.183	0.129
5(4)	0.50	4.0	1.0	0.10	0.262	0.256	0.287	0.237
5(4)	0.50	4.0	1.0	0.50	0.676	0.686	0.695	0.657
3(4)	0.50	4.0	1.0	1.00	0.935	0.870	0.871	1.0
5(4)	2.25	0.6	0.0	0.05	0.311	0.312	0.311	0.304
5(4)	2.25	0.6	0.0	0.10	0.467	0.464	0.465	0.461
3(2)	2.25	0.6	0.0	0.50	0.908	0.890	0.902	0.889
5(4)	2.25	2.0	0.0	0.05	0.214	0.225	0.213	0.215
5(4)	2.25	2.0	0.0	0.10	0.338	0.346	0.337	0.338
5(4)	2.25	2.0	0.0	0.50	0.774	0.776	0.774	0.774
5(4)	2.25	2.0	0.0	1.00	0.929	0.927	0.928	0.927
5(4)	2.25	2.0	0.2	0.05	0.178	0.184	0.176	0.178
5(4)	2.25	2.0	0.2	0.10	0.290	0.294	0.289	0.289
5(4)	2.25	2.0	0.2	0.50	0.723	0.723	0.722	0.722
5(4)	2.25	2.0	0.2	1.00	0.897	0.896	0.896	0.896
5(4)	2.25	2.0	1.0	0.10	0.183	0.185	0.183	0.183
4(4)	2.25	2.0	1.0	0.50	0.564	0.565	0.564	0.564
5(4)	2.75	0.6	0.0	0.05	0.331	0.318	0.334	0.315
5(4)	2.75	0.6	0.0	0.10	0.488	0.473	0.491	0.472
5(4)	2.75	0.6	0.0	0.50	0.952	0.904	0.959	0.920
3(2)	2.75	0.6	0.0	1.00	1.029	0.986	1.018	0.987
5(4)	2.75	0.6	0.2	0.05	0.247	0.242	0.248	0.241
5(4)	2.75	0.6	0.2	0.10	0.390	0.385	0.390	0.384
5(4)	2.75	0.6	0.2	0.50	0.851	0.845	0.851	0.845
3(2)	2.75	0.6	0.2	1.00	0.983	0.966	0.978	0.966
5(4)	2.75	0.6	1.0	0.05	0.124	0.123	0.125	0.123
5(4)	2.75	0.6	1.0	0.10	0.220	0.219	0.220	0.219
5(4)	2.75	0.6	1.0	0.50	0.650	0.648	0.650	0.648
3(2)	2.75	0.6	1.0	1.00	0.853	0.852	0.855	0.852
5(4)	7.50	0.6	0.0	0.10	0.612	0.508	0.612	0.506
5(4)	7.50	0.6	0.0	0.50	1.09	0.939	1.048	0.941
5(4)	7.50	0.6	0.0	1.00	1.10	0.995	1.105	0.998
5(4)	7.50	0.6	0.2	0.10	0.451	0.409	0.441	0.408
5(4)	7.50	0.6	0.2	0.50	0.931	0.887	0.924	0.887
5(4)	7.50	0.6	0.2	1.00	1.029	0.963	1.022	0.985
5(4)	7.50	0.6	1.0	0.10	0.233	0.227	0.230	0.226
5(4)	7.50	0.6	1.0	0.50	0.693	0.686	0.691	0.686
5(4)	7.50	0.6	1.0	1.00	0.900	0.893	0.897	0.893

The exchanger effectiveness is intimately related to C_o (in Stein's notation) by

$$\epsilon = 1 + \frac{1-H}{H} C_o, \quad H < 1 \tag{1}$$

$$\epsilon = H - (H-1)C_o, \quad H > 1 \tag{2}$$

Expressions for C_o can be obtained at each end of the exchanger. Thus at $Z = 0$

$$C_o = \frac{H\bar{\xi}_2(0)}{H-1} \tag{3}$$

and at $z = Z$

$$C_o = \frac{H - \bar{\xi}_1(Z)}{H-1} \tag{4}$$

To compute C_o and hence the effectiveness from the Argonne procedure, we added Equations (3) and (4). Since $\bar{\xi}_2(0)$ and $\bar{\xi}_1(Z)$ depend on the higher-order expansion coefficients, the effectiveness also depends on the higher-order coefficients. Table 1 gives the exchanger effectiveness as obtained by the Argonne procedure, ϵ_A, and by the Nunge-Gill procedure, ϵ_{N-G}. In the table, M indicates the number of nonzero eigenvalues used to calculate ϵ_{N-G}; the number in brackets in the M column is the number of nonzero eigenvalues used to compute ϵ_A. To conform with the Argonne procedure, we have reported ϵ_A only for even values of M. It is clear from the table that except for the two extremes of H, $H = 0.2$ and $H = 7.5$, at small values of the parameter Z or L, the two procedures give effectively the same result.

Also reported in Table 1 are the two values of the effectiveness, ϵ_o and ϵ_L, which can be obtained by using Equations (3) and (4) individually. An overall heat balance indicates that these two values must be the same or else the heat balance will not be satisfied. Therefore, the agreement between these two values as successive terms is included and can serve as an alternate check on the convergence of the series. The reported values of ϵ_o and ϵ_L were calculated for M in the same way as for ϵ_{N-G}. Except for the two extreme values of H again, ϵ_o, ϵ_L, ϵ_{N-G}, and ϵ_A are effectively the same.

Table 2. Convergence of Effectiveness for ($H = 2.25$, $K = 2.0$, $K_w = 0$)

M	ϵ_o	ϵ_L
	$L = 0.05$	
2	0.216	0.198
3	0.200	0.203
4	0.216	0.213
5	0.213	0.215
	$L = 0.1$	
2	0.340	0.323
3	0.327	0.329
4	0.340	0.337
5	0.337	0.338
	$L = 0.5$	
2	0.787	0.771
3	0.773	0.772
4	0.777	0.774
5	0.774	0.774

Stein has suggested that the Nunge-Gill procedure is weakest in the plug flow case for values of KH which are very different from unity. To illustrate the convergence of the effectiveness as successive terms, Table 2 gives values of ϵ_0 and ϵ_L for $KH = 4.5$, the maximum KH product of the cases investigated previously. It is clear from Table 2 that ϵ_0 and ϵ_L are the same and unchanging for $M = 5$ and that the converged value agrees with ϵ_{N-G} in Table 1. Hence, $KH = 4.5$ is not sufficiently different from unity to reflect any significant differences between the computational procedures.

As is almost always the case, it is necessary, when one applies the general theory to a particular physical problem, to examine the related numerical calculations carefully. For the laminar flow case we did this originally by checking the internal consistency and by generating a finite-difference solution for comparison. We have done it again by using an alternate numerical scheme for solving the same linear system of equations. Now we are even more confident that our results for the laminar flow case are accurate enough for the practical purpose of calculating effectiveness, local temperature distributions, and Nusselt numbers. However, it is clear that when one attacks a different physical problem, the numerical techniques employed will have to be tailored to fit the needs of that particular problem.

A different method of solving the countercurrent problem, which employs the Duhamel theorem, has been described briefly by Blanco and Gill (5). In this case the problem is reduced to solving a Fredholm equation of the first kind.

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A Method of Predicting Boiling Pressure Drop for Alkali Metals

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Recently, some alkali-metal boiling pressure-drop data were shown to agree with the constant-slip model boiling pressure-drop correlation of Stone and Damman (*1*). The model assumed in developing the correlation was similar to that of Thom (*2*), in that the slip ratio (mean gas velocity divided by mean liquid velocity) is assumed to be a function of the liquid-to-gas density ratio only. Thom indicates that his method follows the lines suggested by Martinelli and Nelson (*3*) for a horizontal tube, but it has been extended to include the vertical-flow case, whereas Lockhart and Martinelli (*4*) do not include gravitational head effects.

Some modifications of Thom's method (*2*) (based on high pressure, water boiling data) are required in order to correlate low pressure, water boiling data (*1*). The correlation was based on data of the authors (*1*), and of Dengler (*5*) for low pressure water boiling in vertical upflow heat exchangers and on some unpublished NASA sodium boiling data (liquid-to-vapor density ratios from about 800 to 6,000). After the development of this correlation, additional NASA sodium boiling data was obtained, and Peterson (*6*) presented the results of an extensive potassium boiling investigation. The agreement between these data and the correlation of reference *1* suggests that this correlation may be useful in predicting pressure drop for alkali-metal boilers.

ANALYSIS

Application of the conservation of energy, mass, and momentum yields the pressure drop as the sum of three terms: inertial, frictional, and gravitational. The results of Thom (*2*) for constant heat flux, friction factor, slip ratio, and physical properties are given below:

$$\Delta P_I = \frac{G^2}{144\rho_l g_c}\left\{\left[1 + x_E\left(\frac{1}{V}\frac{\rho_l}{\rho_g} - 1\right)\right]\right.$$
$$\left.[1 + x_E(V-1)] - 1\right\} \quad (1)$$

$$\Delta P_F = \frac{f_{TP}G^2}{144\rho_l g_c}\left\{\left[1 + x_E\left(\frac{1}{V}\frac{\rho_l}{\rho_g} - 1\right)\right]\right.$$
$$\left.[1 + x_E(V-1)] + 1\right\}\frac{L}{D} \quad (2)$$

$$\Delta P_G = \frac{\rho_l L}{144}\left(\frac{g}{g_c}\right)\left\{\frac{\frac{1}{V}\left(\frac{\rho_l}{\rho_g} - 1\right)}{\left(\frac{1}{V}\frac{\rho_l}{\rho_g} - 1\right)^2 x_E}\right.$$
$$\left.\ln\left[1 + x_E\left(\frac{1}{V}\frac{\rho_l}{\rho_g} - 1\right)\right] + \frac{\frac{1}{V} - 1}{\frac{1}{V}\frac{\rho_l}{\rho_g} - 1}\right\} \quad (3)$$

The inertial pressure drop ΔP_I is a function of the mean density and velocity at the inlet and exit, and is indepen-